

The Possible $J^{PC} = 0^{+-}$ Exotic State

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We study the possible exotic states with $J^{PC} = 0^{+-}$ using the tetraquark interpolating currents with the QCD sum rule approach. The extracted masses are around 4.85 GeV for the charmonium-like states and 11.25 GeV for the bottomonium-like states. There is no working region for the light tetraquark currents, which implies the light 0^{+-} state may not exist below 2 GeV.

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I. INTRODUCTION

Up to now most of the hadrons observed experimentally can be interpreted as the $q\bar{q}/qqq$ states in the quark model [1, 2]. However there has accumulated some evidence of the exotic state with $J^{PC} = 1^{-+}$ [3–5]. Such a quantum number is not accessible for a conventional meson composed of a pair of quark and anti-quark in the non-relativistic quark model. Sometimes these states are named as exotic states although all the J^{PC} quantum numbers are allowed in QCD.

For a neutral quark model $q\bar{q}$ state, $J = 0$ ensures $L = S$ hence $C = (-)^{L+S} = +1$. Therefore, there exist two possible exotic states with $J^{PC} = 0^{--}$ and 0^{+-} . It's also interesting to note that the J^{PC} quantum number of the local operators composed of a pair of the gluon field strength tensor is either 0^{++} or 0^{-+} .

On the other hand, the tetraquark operators may carry the 0^{--} and 0^{+-} quantum numbers. In fact, the 0^{--} state was investigated systematically using the tetraquark currents with the QCD sum rule method [6, 7]. As a byproduct, it was noted that there does not exist any tetraquark interpolating current without derivative for the $J^{PC} = 0^{+-}$ case.

With the similar formalism, one may construct the possible 0^{+-} tetraquark current by introducing derivatives. There are two kinds of constructions either with the qq basis or with the $\bar{q}q$ basis: $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$. However, they can be related to each other by the Fierz transformation [6]. In this work we use the first set. With these independent 0^{+-} currents, we perform the QCD sum rule analysis and extract the masses of the corresponding currents.

This paper is organized as follows. In Sec. II, we construct the tetraquark currents with $J^{PC} = 0^{+-}$ using the diquark (qq) and antidiquark ($\bar{q}\bar{q}$) fields. In Sec. III, we calculate the correlation functions and spectral densities of the interpolating currents and collect them in the Appendix B. We perform the numerical analysis and extract the masses in Sec. IV for the light and heavy systems respectively. The last section is a brief summary.

II. TETRAQUARK INTERPOLATING CURRENTS

It was shown that the $J^{PC} = 0^{+-}$ tetraquark interpolating currents without derivatives do not exist [6]. So in this work we construct the 0^{+-} currents with the derivatives following the similar steps as in Ref. [6]. We first construct

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two independent tetraquark fields:

$$\begin{aligned} A_{abcd} &= (q_{1a}^T C \gamma^\mu q_{2b}) (\bar{q}_{3c} \overleftrightarrow{D}_\mu C \bar{q}_{4d}^T), \\ P_{abcd} &= (q_{1a}^T C \gamma^\mu \gamma_5 q_{2b}) (\bar{q}_{3c} \overleftrightarrow{D}_\mu \gamma_5 C \bar{q}_{4d}^T). \end{aligned} \quad (1)$$

where q_{1-4} represents the flavor of quarks, and $a-d$ stands for the color indices, $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$, $\overrightarrow{D}_\mu = \overrightarrow{\partial}_\mu + ig A_\mu^a t^a$. It is understood that the index c is the color index of $(\bar{q} \overleftrightarrow{D}_\mu)_c$. In Eqs.(1) and (2) we have used the shorthand notation to simply the expression.

To compose the color singlet tetraquark currents, the diquark and antidiquark should have the same color and spin symmetries. Therefore the color structure of the tetraquark is either $\mathbf{6} \otimes \bar{\mathbf{6}}$ or $\bar{\mathbf{3}} \otimes \mathbf{3}$, which is denoted by labels **6** and **3** respectively. Details can be found in Ref. [6]. Considering both the color and Lorentz structures, we can obtain the currents with $J^{PC} = 0^{+-}$:

$$\begin{aligned} \eta_1(x) &= q_{1a}^T C \gamma^\mu q_{2b} (\bar{q}_{1a} \overleftrightarrow{D}_\mu C \bar{q}_{2b}^T + \bar{q}_{1b} \overleftrightarrow{D}_\mu C \bar{q}_{2a}^T) - q_{1a}^T C \overleftrightarrow{D}_\mu q_{2b} (\bar{q}_{1a} \gamma^\mu C \bar{q}_{2b}^T + \bar{q}_{1b} \gamma^\mu C \bar{q}_{2a}^T), \\ \eta_2(x) &= q_{1a}^T C \gamma^\mu q_{2b} (\bar{q}_{1a} \overleftrightarrow{D}_\mu C \bar{q}_{2b}^T - \bar{q}_{1b} \overleftrightarrow{D}_\mu C \bar{q}_{2a}^T) - q_{1a}^T C \overleftrightarrow{D}_\mu q_{2b} (\bar{q}_{1a} \gamma^\mu C \bar{q}_{2b}^T - \bar{q}_{1b} \gamma^\mu C \bar{q}_{2a}^T), \\ \eta_3(x) &= q_{1a}^T C \gamma^\mu \gamma_5 q_{2b} (\bar{q}_{1a} \overleftrightarrow{D}_\mu \gamma_5 C \bar{q}_{2b}^T + \bar{q}_{1b} \overleftrightarrow{D}_\mu \gamma_5 C \bar{q}_{2a}^T) - q_{1a}^T C \overleftrightarrow{D}_\mu \gamma_5 q_{2b} (\bar{q}_{1a} \gamma^\mu \gamma_5 C \bar{q}_{2b}^T + \bar{q}_{1b} \gamma^\mu \gamma_5 C \bar{q}_{2a}^T), \\ \eta_4(x) &= q_{1a}^T C \gamma^\mu \gamma_5 q_{2b} (\bar{q}_{1a} \overleftrightarrow{D}_\mu \gamma_5 C \bar{q}_{2b}^T - \bar{q}_{1b} \overleftrightarrow{D}_\mu \gamma_5 C \bar{q}_{2a}^T) - q_{1a}^T C \overleftrightarrow{D}_\mu \gamma_5 q_{2b} (\bar{q}_{1a} \gamma^\mu \gamma_5 C \bar{q}_{2b}^T - \bar{q}_{1b} \gamma^\mu \gamma_5 C \bar{q}_{2a}^T). \end{aligned} \quad (2)$$

III. QCD SUM RULE

Consider the two-point correlation function in the framework of QCD sum rule

$$\Pi(q^2) \equiv i \int d^4x e^{iqx} \langle 0 | T \eta(x) \eta^\dagger(0) | 0 \rangle, \quad (3)$$

where η is an interpolating current. At the hadron level, the correlation function $\Pi(p^2)$ is expressed via the dispersion relation:

$$\Pi(p^2) = \int_0^\infty \frac{\rho(s)}{s - p^2 - i\epsilon} ds, \quad (4)$$

where

$$\begin{aligned} \rho(s) &\equiv \sum_n \delta(s - m_n^2) \langle 0 | \eta | n \rangle \langle n | \eta^\dagger | 0 \rangle \\ &= f_X^2 \delta(s - m_X^2) + \text{continuum}, \end{aligned} \quad (5)$$

where m_X is the mass of the resonance X and f_X is the decay constant of the meson:

$$\langle 0 | \eta | X \rangle = f_X. \quad (6)$$

The correlation function can also be calculated at the quark-gluon level using the QCD operator product expansion (OPE) method. It is convenient to evaluate the Wilson coefficient in the coordinate space for the light quark systems and in the momentum space for the heavy quark systems respectively. In our calculation we consider the first order perturbative and various condensates contributions. In order to calculate the gluonic condensate, it is convenient to work in the fixed-point gauge. The massive quark propagator $iS(x, y)$ in an external field in the fixed-point gauge is listed in Appendix A. The quark lines attached with gluon contain terms proportional to y , which we can ignore in the current without derivatives. We keep these terms throughout the evaluation and let y go to zero only after finishing the derivatives. The $\Pi(p^2)$ can be written as:

$$\Pi^{OPE}(p^2) = \int_{4(m_1+m_2)^2}^\infty ds \frac{\rho^{OPE}(s)}{s - p^2 - i\epsilon}, \quad (7)$$

where the m_1 and m_2 are the mass of the quark q_1 and q_2 respectively. In order to suppress the higher state contributions, we perform the Borel transformation to the correlation function, which improves the convergence of the OPE series. With the quark-hadron duality, we obtain:

$$\Pi(M_B^2) = f_X^2 e^{-m_X^2/M_B^2} = \int_{4(m_1+m_2)^2}^\infty ds e^{-s/M_B^2} \rho^{OPE}(s), \quad (8)$$

where s_0 is the threshold parameter and M_B is the Borel parameter. We can extract the meson mass m_X :

$$m_X^2 = -\frac{\frac{\partial}{\partial(1/M_B^2)}\Pi(M_B^2)}{\Pi(M_B^2)} = \frac{\int_{4(m_1+m_2)^2}^{s_0} ds e^{-s/M_B^2} s \rho(s)}{\int_{4(m_1+m_2)^2}^{s_0} ds e^{-s/M_B^2} \rho(s)} \quad (9)$$

For all the tetraquark currents in Eq.(2), we collect the spectral density $\rho^{OPE}(s)$ in the Appendix. The quark condensate $\langle \bar{q}q \rangle$ vanishes due to the special Lorenz structures of the currents. For the $q = u, d$, we do the calculation in the chiral limit $m_q = 0$. Since the contribution of the three gluon condensate $\langle g^2 f G G G \rangle$ is very small, we consider only the power corrections from the following condensates $\langle g^2 G G \rangle$, $\langle \bar{q}g\sigma \cdot Gq \rangle$, $\langle \bar{q}q \rangle^2$ and $\langle \bar{q}g\sigma \cdot Gq \rangle \langle \bar{q}q \rangle$. We list several typical Feynman diagrams in the Fig. 1.

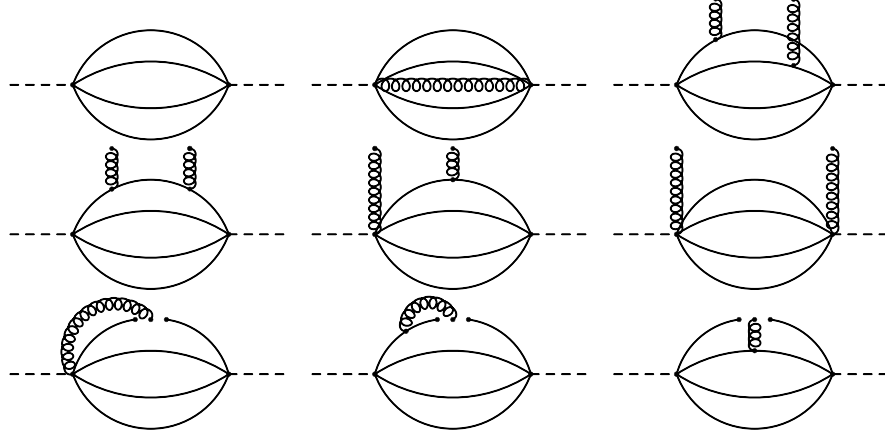


FIG. 1: Some typical Feynman diagrams of the correlation fuctions.

IV. NUMERICAL ANALYSIS

In the QCD sum rule analysis, we use the following values of the quark masses, coupling constant and various condensates [1, 8–10]:

$$\begin{aligned} m_c(m_c) &= (1.23 \pm 0.09) \text{ GeV}, \\ m_b(m_b) &= (4.20 \pm 0.07) \text{ GeV}, \\ \langle \bar{q}q \rangle &= -(0.23 \pm 0.03)^3 \text{ GeV}^3, \\ \langle \bar{q}g\sigma \cdot Gq \rangle &= -M_0^2 \langle \bar{q}q \rangle, \\ M_0^2 &= (0.8 \pm 0.2) \text{ GeV}, \\ \langle \bar{s}s \rangle / \langle \bar{q}q \rangle &= 0.8 \pm 0.2, \\ \langle g^2 G G \rangle &= (0.88 \pm 0.13) \text{ GeV}^4, \\ \alpha_s(1.7 \text{ GeV}) &= 0.328 \pm 0.03 \pm 0.025. \end{aligned} \quad (10)$$

The Borel mass M_B and the threshold value s_0 are two pivotal parameters. The working region of the Borel mass is determined by the convergence of the OPE and the pole contribution. The requirement of the convergence of the OPE determines the lower bound M_{Bmin} of the Borel mass, and the pole contribution determines the upper bound M_{Bmax} .

In this work, there is no contribution from the quark condensate $\langle \bar{q}q \rangle$. The correction from the condensate $\langle \bar{q}g\sigma \cdot Gq \rangle \langle \bar{q}q \rangle$ is the most important numerically. Its contribution is bigger than that from the gluon condensate $\langle g^2 G G \rangle$, mixed condensate $\langle \bar{q}g\sigma \cdot Gq \rangle$ and the four quark condensate $\langle \bar{q}q \rangle^2$. The mixed condensate is also very important numerically for the currents $\eta_{1,3}$. We require that the condensate $\langle \bar{q}g\sigma \cdot Gq \rangle \langle \bar{q}q \rangle$ be less than one ninth of the perturbative term to ensure the convergence of OPE, which leads to the lower bound of the Borel parameter working window.

	Current	$s_0(\text{GeV}^2)$	$[M_{Bmin}, M_{Bmax}](\text{GeV})$	$M_B(\text{GeV})$	$m_X(\text{GeV})$	PC(%)
$q_1, q_2 = u, d$	η_{1-4}^q	-	-	-	-	-
$q_1 = u, d$ $q_2 = c$	η_1^c	27	$1.8 \sim 2.1$	2.0	4.76 ± 0.08	37.4
	η_2^c	28	$1.8 \sim 2.1$	2.0	4.85 ± 0.09	39.9
	η_3^c	29	$1.8 \sim 2.1$	2.0	4.96 ± 0.13	42.4
	η_4^c	28	$1.8 \sim 2.1$	2.0	4.83 ± 0.07	40.9
$q_1 = u, d$ $q_2 = b$	η_1^b	140	$2.9 \sim 3.3$	3.1	11.24 ± 0.17	52.2
	η_2^b	142	$2.9 \sim 3.3$	3.1	11.27 ± 0.14	55.6
	η_3^b	142	$2.9 \sim 3.3$	3.1	11.30 ± 0.17	55.0
	η_4^b	142	$2.9 \sim 3.3$	3.1	11.27 ± 0.09	55.5

TABLE I: The threshold values, Borel window, Borel parameter for the different tetraquark currents.

The pole contribution (PC) is defined as

$$\text{PC} = \frac{\int_{4(m_1+m_2)^2}^{s_0} ds e^{-s/M_B^2} \rho(s)}{\int_{4(m_1+m_2)^2}^{\infty} ds e^{-s/M_B^2} \rho(s)} \quad (11)$$

which depends on both the Borel mass M_B and the threshold value s_0 . s_0 is chosen around the region where the variation of m_X with M_B is minimum. Requiring the PC be larger than 30% \sim 50%, we get the upper bound M_{Bmax} of the Borel mass M_B . We list the working region of the Borel parameter for the four currents with different quark composition in Table I. For the η_{1-4}^c , we get the upper bound of Borel parameter M_B by requiring the PC be larger than 30%. For η_{1-4}^b we require the PC be larger than 40%. The masses are extracted using the threshold values s_0 and Borel parameters M_B listed in Table I. The last column is the pole contribution with the corresponding s_0 and M_B .

For the light tetraquark systems, there does not exist a working region for the sum rules. Even in the extreme case that the pole contribution is \sim 30% and the contribution of the condensate $\langle \bar{q}g\sigma \cdot Gq \rangle \langle \bar{q}q \rangle$ is around the leading order contribution, the lower bound M_{Bmin} is still much larger than the upper bound M_{Bmax} . In other words, there is no working region for light quark systems. As shown in Fig.(2,3), the extracted mass grows monotonically with s_0 which implies the 0^{+-} state does not exist below 2 GeV. We note that the light $J^{PC} = 0^{--}$ state does not exist either [6]. The 0^{+-} and 0^{-+} channels are in strong contrast with the 0^{++} case, where there exist stable tetraquark QCD sum rules and the extracted scalar meson masses agree with the experimental scalar spectrum nicely [11].

For the heavy systems, the variation of m_X with s_0 and M_B is presented in Figs. (4)-(11). All the sum rules are very stable with reasonable variations of s_0 and M_B . The presence of the two heavy quarks reduces the kinetic energy of the tetraquark system, hence helps to stabilize the sum rules. Numerically, the masses of the 0^{+-} states are slightly larger than those of the 0^{--} states [7].

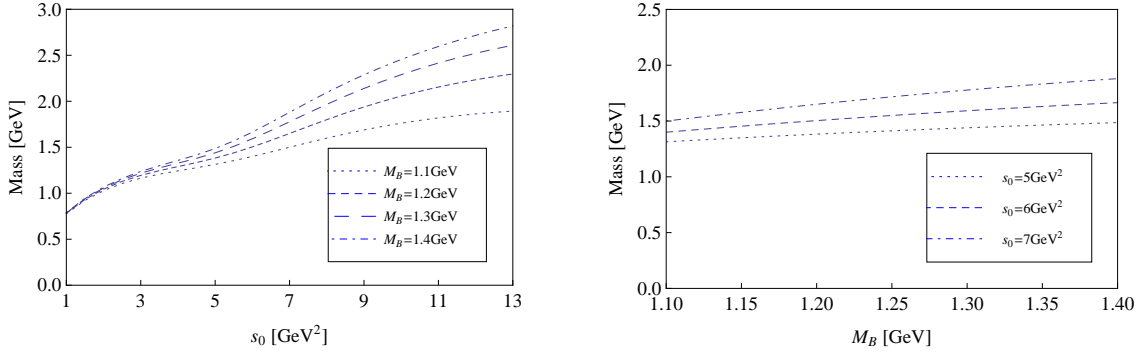
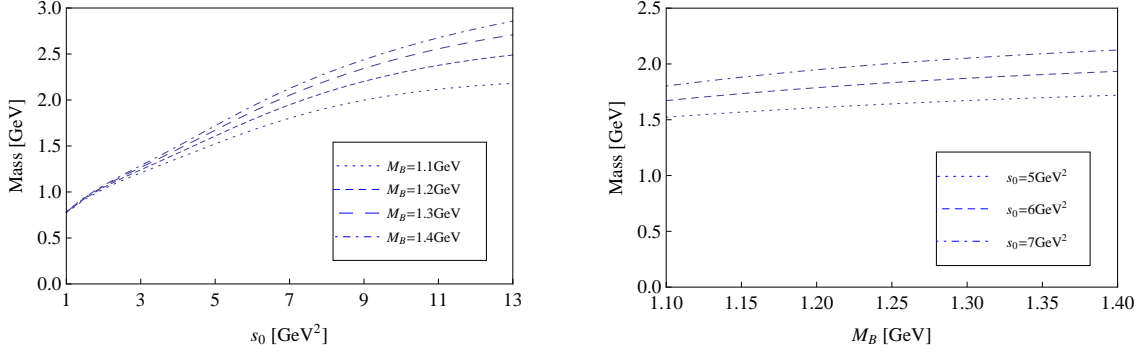
V. SUMMARY

The exotic state with $J^{PC} = 0^{+-}$ cannot be composed of a pair of quark and anti-quark. In order to explore these exotic states, we have constructed four tetraquark interpolating operators. Then we make the operator product expansion and extract the spectral density. Because of the special Lorentz structures of the currents, the quark condensate $\langle \bar{q}q \rangle$ vanishes.

For the light tetraquark systems, there does not exist a working region of the Borel parameter and threshold for all the derived sum rules. It seems that none of these independent interpolating currents supports a resonant signal below 2 GeV, which is consistent with the current experimental data [1]. In contrast, there exist very stable QCD sum rules constructed from the tetraquark interpolating operators in the scalar channel. The extracted scalar spectrum agrees with the experimental data nicely [11].

For the heavy quark systems, the 0^{+-} tetraquark sum rules are quite stable. The presence of two heavy quarks may render the kinetic energy of the tetraquark system, which is helpful in the formation of bound states. The extracted masses from the four interpolating currents η_{1-4}^c are around 4.76 \sim 4.96 GeV for the charmonium-like states. For the bottomonium-like 0^{+-} states, their masses are about 11.2 \sim 11.3 GeV. It's very interesting to note that the mass of the 0^{+-} charmonium-like state extracted from the tetraquark sum rules is numerically quite close to the mass of the 0^{+-} hybrid charmonium extracted on the lattice [12, 13].

Because of the special "exotic" quantum number, the 0^{+-} charmonium-like state does not decay into a pair of particle (H) and anti-particle (\bar{H}). There are two types of 0^{+-} states with different isospin and G-parity: $I^G = 0^-$

FIG. 2: The variation of m_X with M_B (left) and s_0 (right) for the current η_1^q .FIG. 3: The variation of m_X with M_B (left) and s_0 (right) for the current η_2^q .

I^G	S-wave	P-wave
0^-	$\chi_{c1}(1P)h_1(1170) \dots$	$D^0(1865)\bar{D}_1(2420)^0 + c.c., D^*(2007)^0\bar{D}_0^*(2400)^0 + c.c.,$ $\eta_c(1S)h_1(1170), J/\psi(1S)f_0(600),$ $J/\psi f_0(980), J/\psi f_1(1285), \chi_{c0}(1P)\omega(782), \chi_{c1}(1P)\omega(782),$ $\psi(2S)f_0(600), \psi(3770)f_0(600) \dots$
1^+	$J/\psi(1S)\pi_1(1400),$ $J/\psi(1S)\pi_1(1600),$ $\chi_{c1}(1P)b_1(1235) \dots$	$D^0(1865)\bar{D}_1(2420)^0 + c.c., D^*(2007)^0\bar{D}_0^*(2400)^0 + c.c.,$ $D^*(2007)^0\bar{D}_1(2420)^0 + c.c.,$ $\eta_c(1S)b_1(1235), J/\psi(1S)a_0(980), J/\psi(1S)a_1(1260),$ $\chi_{c0}(1P)\rho(770), \chi_{c1}(1P)\rho(770) \dots$

TABLE II: The possible decay modes of the 0^{+-} charmonium-like state.

and $I^G = 1^+$. Only a few S-wave decay modes are allowed. Some possible two-body decay modes are listed in Table II. Replacing the D meson by B meson, one gets the decay patterns of the bottomonium-like states so long as the kinematics allows. The 0^{+-} state may be searched for experimentally at facilities such as Super-B factories, PANDA, LHC and RHIC in the future, especially at RHIC and LHC where plenty of charm, anti-charm and light quarks are produced.

Acknowledgments

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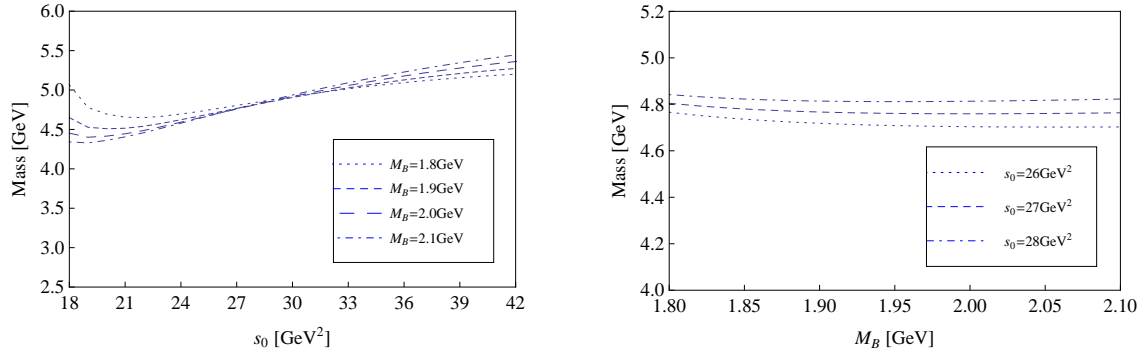


FIG. 4: The variation of m_X with M_B (left) and s_0 (right) for the current η_1^c .

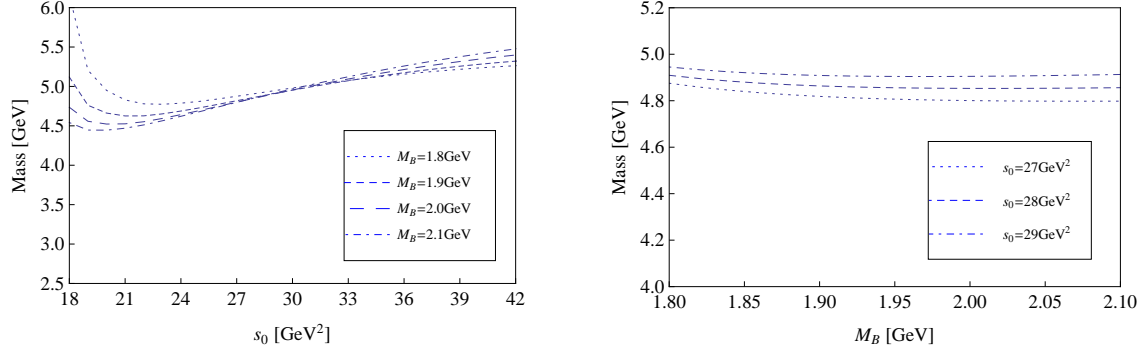


FIG. 5: The variation of m_X with M_B (left) and s_0 (right) for the current η_2^c .

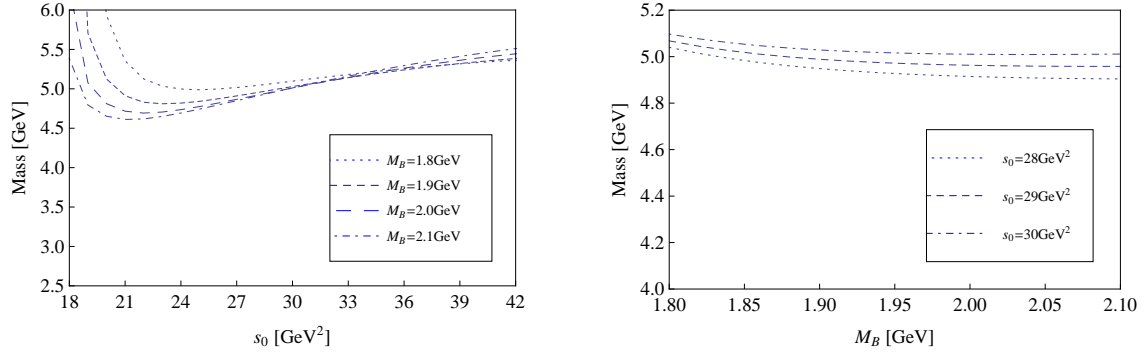


FIG. 6: The variation of m_X with M_B (left) and s_0 (right) for the current η_3^c .

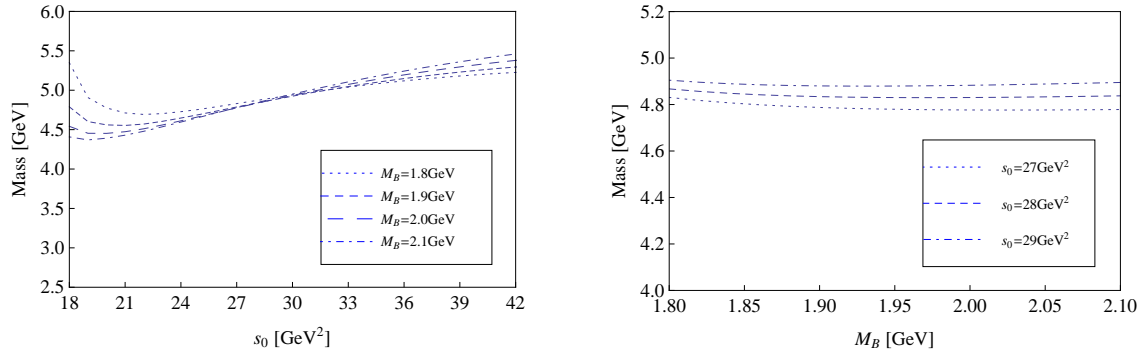


FIG. 7: The variation of m_X with M_B (left) and s_0 (right) for the current η_4^c .

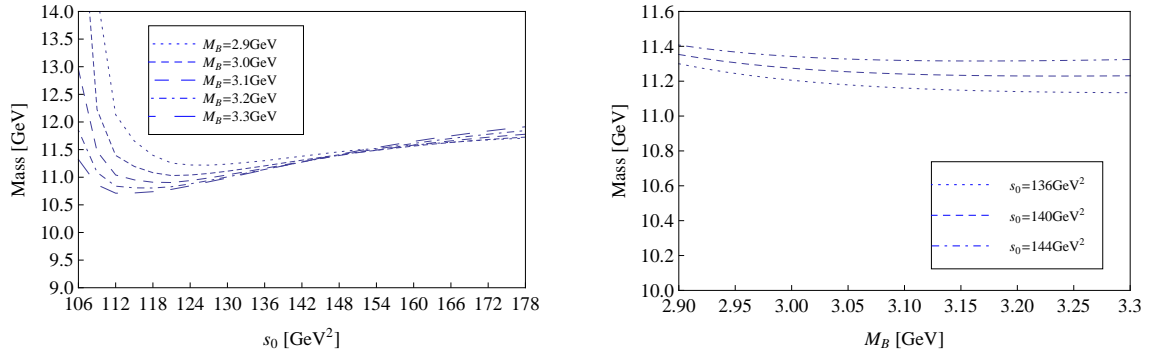


FIG. 8: The variation of m_X with M_B (left) and s_0 (right) for the current η_1^b .

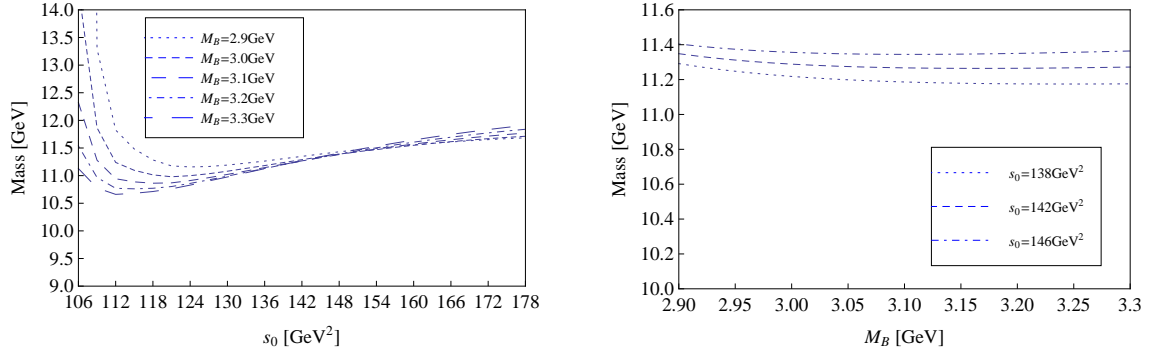


FIG. 9: The variation of m_X with M_B (left) and s_0 (right) for the current η_2^b .

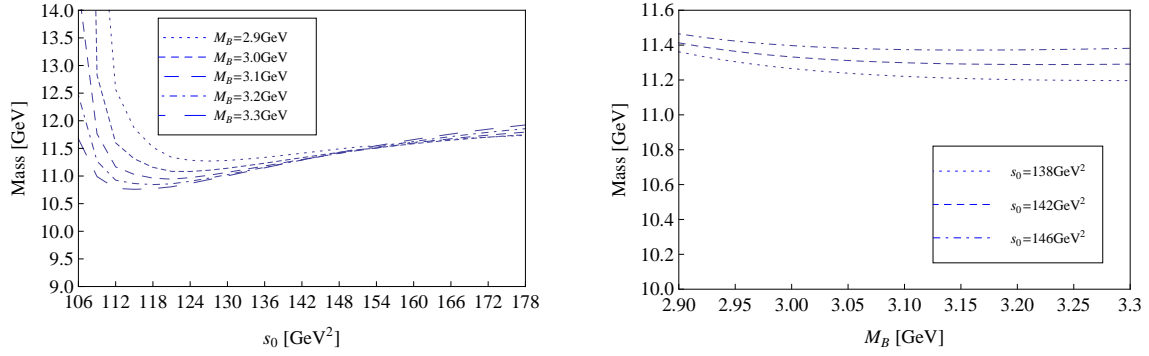


FIG. 10: The variation of m_X with M_B (left) and s_0 (right) for the current η_3^b .

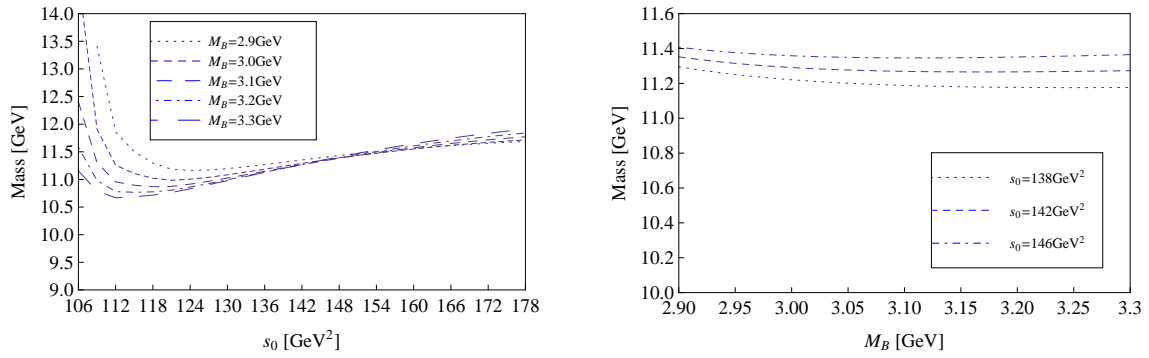


FIG. 11: The variation of m_X with M_B (left) and s_0 (right) for the current η_4^b .

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Appendix A: The Momentum Space Propagator

The fixed-point gauge is defined as:

$$(x - x_0)^\mu A_\mu^a(x) = 0 \quad (\text{A1})$$

where x_0 is an arbitrary point in space which can be chosen to be the origin. Then the potential A_μ^a can be expressed in terms of the field strength tensor $G_{\mu\nu}$ ($G_{\mu\nu} = \frac{\lambda^a}{2} G_{\mu\nu}^a$) [14, 15]:

$$\begin{aligned} A_\mu(x) = & \int_0^1 t dt G_{\nu\mu}(tx) x^\nu \\ & \frac{1}{2} x^\nu G_{\nu\mu}(0) + \frac{1}{3} x^\alpha x^\nu D_\alpha G_{\nu\mu}(0) + \frac{1}{8} x^\alpha x^\beta x^\nu D_\alpha D_\beta G_{\nu\mu}(0) + \dots, \end{aligned} \quad (\text{A2})$$

Denote the massive quark propagator between the position x and y in the coordinate space as $iS(x, y)$. The massive quark propagator in the momentum space is [14]:

$$iS(p) = iS_0(p) + iS_g(p) + iS_{gg}(p) + \dots, \quad (\text{A3})$$

where $iS_0(p)$ is the free quark propagator:

$$iS_0(p) = \frac{i}{\hat{p} - m}, \quad (\text{A4})$$

where $\hat{p} = \gamma^\mu p_\mu$, $iS_g(p)$ is the quark propagator with one gluon leg attached:

$$iS_g(p) = \frac{i}{4} \frac{\lambda^n}{2} g_s G_{\mu\nu}^n \frac{\sigma^{\mu\nu}(\hat{p} + m) + (\hat{p} + m)\sigma^{\mu\nu}}{(p^2 - m^2)^2} + \frac{i}{2} \frac{\lambda^n}{2} g_s G_{\mu\nu}^n \left[\frac{2y^\mu p^\nu (\hat{p} + m)}{(p^2 - m^2)^2} - \frac{y^\mu \gamma^\nu}{p^2 - m^2} \right] \quad (\text{A5})$$

$iS_{gg}(p)$ is the quark propagator with two gluon legs attached:

$$\begin{aligned} iS_{gg}(p) = & -\frac{i}{4} \frac{\lambda^a}{2} \frac{\lambda^b}{2} g_s^2 G_{\mu\rho}^a G_{\nu\sigma}^b \frac{\hat{p} + m}{(p^2 - m^2)^5} (f^{\mu\rho\nu\sigma} + f^{\mu\nu\rho\sigma} + f^{\mu\nu\sigma\rho}) \\ & -\frac{i}{4} \frac{\lambda^a}{2} \frac{\lambda^b}{2} g_s^2 G_{\mu\rho}^a G_{\nu\sigma}^b \frac{\hat{p} + m}{(p^2 - m^2)^4} \left[y^\sigma (f^{\mu\rho\nu} + f^{\mu\nu\rho}) + y^\rho f^{\mu\nu\sigma} - i y^\rho y^\sigma (p^2 - m^2) \right] \end{aligned} \quad (\text{A6})$$

where $f^{\mu\nu\dots\alpha\beta} = \gamma^\mu(\hat{p} + m)\gamma^\nu(\hat{p} + m)\dots\gamma^\alpha(\hat{p} + m)\gamma^\beta(\hat{p} + m)$.

Appendix B: The Spectral Densities

In this appendix, we list the spectral densities of the tetraquark interpolating currents. For the light quark systems ($q_1, q_2 = u, d$), the spectral densities are:

$$\begin{aligned}
\rho_1(s) &= \frac{s^5}{51200\pi^6} \left(1 + \frac{17}{108} \frac{\alpha}{\pi}\right) - \frac{\langle g^2 GG \rangle s^3}{18432\pi^6} - \frac{\langle \bar{q}q \rangle^2 s^2}{6\pi^2} - \frac{799 \langle \bar{q}g_s \sigma \cdot Gq \rangle \langle \bar{q}q \rangle s}{768\pi^2} \\
\rho_2(s) &= \frac{s^5}{102400\pi^6} \left(1 + \frac{7}{54} \frac{\alpha}{\pi}\right) + \frac{\langle g^2 GG \rangle s^3}{18432\pi^6} - \frac{\langle \bar{q}q \rangle^2 s^2}{12\pi^2} - \frac{245 \langle \bar{q}g_s \sigma \cdot Gq \rangle \langle \bar{q}q \rangle s}{768\pi^2} \\
\rho_3(s) &= \frac{s^5}{51200\pi^6} \left(1 + \frac{17}{108} \frac{\alpha}{\pi}\right) - \frac{\langle g^2 GG \rangle s^3}{18432\pi^6} - \frac{\langle \bar{q}q \rangle^2 s^2}{6\pi^2} - \frac{7 \langle \bar{q}g_s \sigma \cdot Gq \rangle \langle \bar{q}q \rangle s}{12\pi^2} \\
\rho_4(s) &= \frac{s^5}{102400\pi^6} \left(1 + \frac{7}{54} \frac{\alpha}{\pi}\right) + \frac{\langle g^2 GG \rangle s^3}{18432\pi^6} - \frac{\langle \bar{q}q \rangle^2 s^2}{12\pi^2} - \frac{7 \langle \bar{q}g_s \sigma \cdot Gq \rangle \langle \bar{q}q \rangle s}{24\pi^2}
\end{aligned} \tag{B1}$$

For the heavy systems ($q_1 = u, d, q_2 = c, d$), the spectral densities are:

$$\rho(s) = \rho^{pert}(s) + \rho^{\langle GG \rangle}(s) + \rho^{\langle \bar{q}Gq \rangle}(s) + \rho^{\langle \bar{q}q \rangle^2}(s) + \rho^{\langle \bar{q}Gq \rangle \langle \bar{q}q \rangle}(s) \tag{B2}$$

For the condensate $\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle$, it contains two parts: one part could be written as ρ , and the other part couldn't, which we perform the Borel transformation directly. Therefore

$$\Pi^{\langle \bar{q}Gq \rangle \langle \bar{q}q \rangle}(M_B^2) = \int_{4m^2}^{\infty} ds e^{-s/M_B^2} \rho^{\langle \bar{q}Gq \rangle \langle \bar{q}q \rangle^1}(s) + \Pi^{\langle \bar{q}Gq \rangle \langle \bar{q}q \rangle^2}(M_B^2). \tag{B3}$$

For the interpolating current η_1 :

$$\begin{aligned}
\rho_1^{pert}(s) &= \frac{384}{\pi^4} \left[(16\rho_{115}^L(s) + m^2\rho_{114}^L(s) - 2m^2\rho_{114}^K(s) + 6m^2\rho_{114}^I(s) - \rho_{114}^O(s) + 2\rho_{114}^N(s) - 4\rho_{114}^J(s)) \right. \\
&\quad \left. + \frac{\alpha}{\pi} \left(\frac{17}{6} \rho_{115}^L(s) - \frac{25}{24} m^2 \rho_{114}^I(s) \right) \right]
\end{aligned} \tag{B4}$$

$$\begin{aligned}
\rho_1^{\langle g^2 GG \rangle}(s) &= \frac{\langle g^2 GG \rangle}{\pi^4} \left[5\rho_{123}^J(s) + \frac{23}{8} \rho_{123}^N(s) - \frac{2}{3} \rho_{213}^N(s) - \frac{10}{3} \rho_{224}^N(s) + \frac{1}{6} \rho_{123}^O(s) + \frac{1}{3} \rho_{213}^O(s) - \frac{10}{3} \rho_{224}^O(s) \right. \\
&\quad - \frac{39}{8} \rho_{123}^I(s) + 1152m^2\rho_{134}^I(s) - 20m^2\rho_{224}^I(s) - 16m^2\rho_{133}^J(s) - 768m^2\rho_{144}^I(s) - \frac{5}{3} \rho_{223}^J(s) \\
&\quad + 192m^2(\rho_{144}^N(s) + \rho_{414}^N(s)) - 96m^2(\rho_{144}^O(s) + \rho_{414}^O(s)) + 1152m^4\rho_{144}^I(s) - \frac{95}{48} \rho_{113}^K(s) + \frac{5}{6} m^2 \rho_{123}^K(s) \\
&\quad - 192m^4(\rho_{144}^K(s) + \rho_{414}^K(s)) + \frac{10}{3} m^2 \rho_{213}^K(s) - \frac{5}{3} \rho_{214}^K(s) - \frac{5}{3} m^2 \rho_{224}^K(s) - 240m^2 \rho_{314}^K(s) + \frac{91}{48} \rho_{113}^L(s) \\
&\quad - \frac{m^2}{2} \rho_{123}^L(s) - \frac{29}{3} \rho_{124}^L(s) + 160m^2 \rho_{134}^L(s) + 3072m^2 \rho_{145}^L(s) + 10m^2 \rho_{224}^L(s) - \frac{31}{12} \rho_{124}^O(s) - \frac{8}{3} \rho_{214}^O(s) \\
&\quad \left. - 192m^2 \rho_{134}^K(s) + 192m^4 \rho_{144}^L(s) \right]
\end{aligned} \tag{B5}$$

$$\begin{aligned}
\rho_1^{\langle \bar{q}Gq \rangle}(s) &= -\frac{m \langle \bar{q}g_s \sigma \cdot Gq \rangle}{3\pi^2} (29\rho_{112}^I(s) + 39m^2\rho_{122}^I(s) - 5\rho_{212}^N(s) - 116\rho_{123}^M(s) - 40\rho_{213}^M(s) + 10m^2\rho_{122}^K(s) + 5m^2\rho_{212}^K(s) \\
&\quad + \frac{521}{8} \rho_{112}^K(s))
\end{aligned} \tag{B6}$$

$$\rho_1^{\langle \bar{q}q \rangle^2}(s) = \frac{16}{3} \langle \bar{q}q \rangle^2 (\rho_{110}^Q(s) - m^4 \rho_{110}^I(s)) \tag{B7}$$

$$\rho_1^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^1}(s) = \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle \left(\frac{1}{18} \rho_{120}^Q(s) - \frac{119}{36} m^2 \rho_{110}^I(s) - \frac{1}{18} m^4 \rho_{120}^I(s) - \frac{2}{3} \rho_{110}^P(s) + \frac{61}{48} \rho_{110}^N(s) + \frac{61}{48} m^2 \rho_{110}^K(s) \right) \tag{B8}$$

$$\Pi_1^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^2}(M_B^2) = \frac{2}{3} \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle (m^4 \Pi^I(M_B^2) - \Pi^{II}(M_B^2)) \tag{B9}$$

For the interpolating current η_2 :

$$\rho_2^{pert}(s) = \frac{192}{\pi^4} \left[(16\rho_{115}^L(s) + m^2\rho_{114}^L(s) - 2m^2\rho_{114}^K(s) + 6m^2\rho_{114}^I(s) - \rho_{114}^O(s) + 2\rho_{114}^N(s) - 4\rho_{114}^J(s)) \right. \\ \left. + \frac{\alpha}{\pi} \left(\frac{7}{6}\rho_{115}^L(s) - \frac{5}{24}m^2\rho_{114}^I(s) \right) \right] \quad (B10)$$

$$\rho_2^{\langle g^2 GG \rangle}(s) = \frac{\langle g^2 GG \rangle}{\pi^4} \left[-3\rho_{123}^J(s) + \frac{7}{8}\rho_{123}^N(s) + \frac{2}{3}\rho_{213}^N(s) - \frac{2}{3}\rho_{224}^N(s) - \frac{7}{6}\rho_{123}^O(s) - \frac{1}{3}\rho_{213}^O(s) - \frac{1}{3}\rho_{224}^O(s) \right. \\ + \frac{57}{8}m^2\rho_{123}^I(s) + 576m^2\rho_{134}^I(s) - 4m^2\rho_{224}^I(s) - 8m^2\rho_{133}^J(s) - 384m^2\rho_{144}^J(s) - \frac{1}{3}m^2\rho_{223}^J(s) \\ + 96m^2(\rho_{144}^N(s) + \rho_{414}^N(s)) - 48m^2(\rho_{144}^O(s) + \rho_{414}^O(s)) + 576m^4\rho_{144}^I(s) + \frac{17}{48}\rho_{113}^K(s) + \frac{1}{6}m^2\rho_{123}^K(s) \\ - 96m^2\rho_{134}^K(s) - 96m^4(\rho_{144}^K(s) + \rho_{414}^K(s)) - \frac{10}{3}m^2\rho_{213}^K(s) - \frac{1}{3}\rho_{214}^K(s) - \frac{1}{3}m^2\rho_{224}^K(s) \\ - 120m^2\rho_{314}^K(s) + \frac{47}{48}\rho_{113}^L(s) + \frac{3}{2}m^2\rho_{123}^L(s) + \frac{35}{3}\rho_{124}^L(s) + 80m^2\rho_{134}^L(s) + 96m^4\rho_{144}^L(s) \\ \left. + 1536m^2\rho_{145}^L(s) + 2m^2\rho_{224}^L(s) + \frac{1}{12}\rho_{124}^H(s) + \frac{8}{3}\rho_{214}^H(s) \right] \quad (B11)$$

$$\rho_2^{\langle \bar{q}Gq \rangle}(s) = \frac{m\langle \bar{q}g\sigma \cdot Gq \rangle}{3\pi^2} (11\rho_{112}^I(s) + \rho_{212}^N(s) - 44\rho_{123}^M(s) + 8\rho_{213}^M(s) + 9m^2\rho_{122}^I(s) - \frac{73}{8}\rho_{112}^K(s) + 10m^2\rho_{122}^K(s) \\ - m^2\rho_{212}^K(s)) \quad (B12)$$

$$\rho_2^{\langle \bar{q}q \rangle^2}(s) = \frac{8}{3}\langle \bar{q}q \rangle^2(\rho_{110}^Q(s) - m^4\rho_{110}^I(s)) \quad (B13)$$

$$\rho_2^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^1}(s) = \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle \left(\frac{17}{18}\rho_{120}^Q(s) - \frac{79}{36}m^2\rho_{110}^I(s) - \frac{17}{18}m^4\rho_{120}^I(s) - \frac{1}{3}\rho_{110}^P(s) + \frac{29}{48}\rho_{110}^N(s) + \frac{29}{48}m^2\rho_{110}^K(s) \right) \quad (B14)$$

$$\Pi_2^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^2}(M_B^2) = \frac{1}{3}\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle (m^4\Pi^I(M_B^2) - \Pi^{II}(M_B^2)) \quad (B15)$$

For the interpolating current η_3 :

$$\rho_3^{pert}(s) = \frac{384}{\pi^4} \left[(16\rho_{115}^L(s) + m^2\rho_{114}^L(s) - 2m^2\rho_{114}^K(s) + 6m^2\rho_{114}^I(s) - \rho_{114}^O(s) + 2\rho_{114}^N(s) - 4\rho_{114}^J(s)) \right. \\ \left. + \frac{\alpha}{\pi} \left(\frac{17}{6}\rho_{115}^L(s) - \frac{25}{24}m^2\rho_{114}^I(s) \right) \right] \quad (B16)$$

$$\rho_3^{\langle g^2 GG \rangle}(s) = \frac{\langle g^2 GG \rangle}{\pi^4} \left[5\rho_{123}^J(s) + \frac{23}{8}\rho_{123}^N(s) - \frac{2}{3}\rho_{213}^N(s) - \frac{10}{3}\rho_{224}^N(s) + \frac{1}{6}\rho_{123}^O(s) + \frac{1}{3}\rho_{213}^O(s) - \frac{10}{3}\rho_{224}^O(s) \right. \\ - \frac{39}{8}\rho_{123}^I(s) + 1152m^2\rho_{134}^I(s) - 20m^2\rho_{224}^I(s) - 16m^2\rho_{133}^J(s) - 768m^2\rho_{144}^J(s) - \frac{5}{3}\rho_{223}^J(s) \\ + 192m^2(\rho_{144}^N(s) + \rho_{414}^N(s)) - 96m^2(\rho_{144}^O(s) + \rho_{414}^O(s)) + 1152m^4\rho_{144}^I(s) - \frac{95}{48}\rho_{113}^K(s) + \frac{5}{6}m^2\rho_{123}^K(s) \\ - 192m^4(\rho_{144}^K(s) + \rho_{414}^K(s)) + \frac{10}{3}m^2\rho_{213}^K(s) - \frac{5}{3}\rho_{214}^K(s) - \frac{5}{3}m^2\rho_{224}^K(s) - 240m^2\rho_{314}^K(s) + \frac{91}{48}\rho_{113}^L(s) \\ - \frac{m^2}{2}\rho_{123}^L(s) - \frac{29}{3}\rho_{124}^L(s) + 160m^2\rho_{134}^L(s) + 3072m^2\rho_{145}^L(s) + 10m^2\rho_{224}^L(s) - \frac{31}{12}\rho_{124}^O(s) - \frac{8}{3}\rho_{214}^O(s) \\ \left. - 192m^2\rho_{134}^K(s) + 192m^4\rho_{144}^L(s) \right] \quad (B17)$$

$$\rho_3^{\langle \bar{q}Gq \rangle}(s) = \frac{m\langle \bar{q}g\sigma \cdot Gq \rangle}{3\pi^2} (29\rho_{112}^I(s) + 39m^2\rho_{122}^I(s) - 5\rho_{212}^N(s) - 116\rho_{123}^M(s) - 40\rho_{213}^M(s) + 10m^2\rho_{122}^K(s) + 5m^2\rho_{212}^K(s) \\ + \frac{521}{8}\rho_{112}^K(s)) \quad (B18)$$

$$\rho_3^{\langle \bar{q}q \rangle^2}(s) = \frac{16}{3} \langle \bar{q}q \rangle^2 (\rho_{110}^Q(s) - m^4 \rho_{110}^I(s)) \quad (B19)$$

$$\rho_3^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^1}(s) = \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle \left(\frac{1}{18} \rho_{120}^Q(s) - \frac{119}{36} m^2 \rho_{110}^I(s) - \frac{1}{18} m^4 \rho_{120}^I(s) - \frac{2}{3} \rho_{110}^P(s) + \frac{61}{48} \rho_{110}^N(s) + \frac{61}{48} m^2 \rho_{110}^K(s) \right) \quad (B20)$$

$$\Pi_3^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^2}(M_B^2) = \frac{2}{3} \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle (m^4 \Pi^I(M_B^2) - \Pi^{II}(M_B^2)) \quad (B21)$$

For the interpolating current η_4 :

$$\rho_4^{pert}(s) = \frac{192}{\pi^4} \left[(16 \rho_{115}^L(s) + m^2 \rho_{114}^L(s) - 2m^2 \rho_{114}^K(s) + 6m^2 \rho_{114}^I(s) - \rho_{114}^O(s) + 2 \rho_{114}^N(s) - 4 \rho_{114}^J(s)) \right. \\ \left. + \frac{\alpha}{\pi} \left(\frac{7}{6} \rho_{115}^L(s) - \frac{5}{24} m^2 \rho_{114}^I(s) \right) \right] \quad (B22)$$

$$\rho_4^{\langle g^2 GG \rangle}(s) = \frac{\langle g^2 GG \rangle}{\pi^4} \left[-3 \rho_{123}^J(s) + \frac{7}{8} \rho_{123}^N(s) + \frac{2}{3} \rho_{213}^N(s) - \frac{2}{3} \rho_{224}^N(s) - \frac{7}{6} \rho_{123}^O(s) - \frac{1}{3} \rho_{213}^O(s) - \frac{1}{3} \rho_{224}^O(s) \right. \\ + \frac{57}{8} m^2 \rho_{123}^I(s) + 576 m^2 \rho_{134}^I(s) - 4 m^2 \rho_{224}^I(s) - 8 m^2 \rho_{133}^J(s) - 384 m^2 \rho_{144}^J(s) - \frac{1}{3} m^2 \rho_{223}^J(s) \\ + 96 m^2 (\rho_{144}^N(s) + \rho_{414}^N(s)) - 48 m^2 (\rho_{144}^O(s) + \rho_{414}^O(s)) + 576 m^4 \rho_{144}^I(s) + \frac{17}{48} \rho_{113}^K(s) + \frac{1}{6} m^2 \rho_{123}^K(s) \\ - 96 m^2 \rho_{134}^K(s) - 96 m^4 (\rho_{144}^K(s) + \rho_{414}^K(s)) - \frac{10}{3} m^2 \rho_{213}^K(s) - \frac{1}{3} \rho_{214}^K(s) - \frac{1}{3} m^2 \rho_{224}^K(s) \\ - 120 m^2 \rho_{314}^K(s) + \frac{47}{48} \rho_{113}^L(s) + \frac{3}{2} m^2 \rho_{123}^L(s) + \frac{35}{3} \rho_{124}^L(s) + 80 m^2 \rho_{134}^L(s) + 96 m^4 \rho_{144}^L(s) \\ \left. + 1536 m^2 \rho_{145}^L(s) + 2 m^2 \rho_{224}^L(s) + \frac{1}{12} \rho_{124}^H(s) + \frac{8}{3} \rho_{214}^H(s) \right] \quad (B23)$$

$$\rho_4^{\langle \bar{q}Gq \rangle}(s) = -\frac{m \langle \bar{q}g\sigma \cdot Gq \rangle}{3\pi^2} (11 \rho_{112}^I(s) + \rho_{212}^N(s) - 44 \rho_{123}^M(s) + 8 \rho_{213}^M(s) + 9 m^2 \rho_{122}^I(s) - \frac{73}{8} \rho_{112}^K(s) + 10 m^2 \rho_{122}^K(s) \\ - m^2 \rho_{212}^K(s)) \quad (B24)$$

$$\rho_4^{\langle \bar{q}q \rangle^2}(s) = \frac{8}{3} \langle \bar{q}q \rangle^2 (\rho_{110}^Q(s) - m^4 \rho_{110}^I(s)) \quad (B25)$$

$$\rho_4^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^1}(s) = \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle \left(\frac{17}{18} \rho_{120}^Q(s) - \frac{79}{36} m^2 \rho_{110}^I(s) - \frac{17}{18} m^4 \rho_{120}^I(s) - \frac{1}{3} \rho_{110}^P(s) + \frac{29}{48} \rho_{110}^N(s) + \frac{29}{48} m^2 \rho_{110}^K(s) \right) \quad (B26)$$

$$\Pi_4^{\langle \bar{q}q \rangle \langle \bar{q}Gq \rangle^2}(M_B^2) = \frac{1}{3} \langle \bar{q}q \rangle \langle \bar{q}Gq \rangle (m^4 \Pi^I(M_B^2) - \Pi^{II}(M_B^2)) \quad (B27)$$

The functions $\rho_{hjk}^{I,J,K\dots}(s)$ and $\Pi^{I,II}(M_B^2)$ in the above expressions are defined as:

$$\rho_{hjk}^I(s) = \frac{(-1)^k 4^{-k-2}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(3-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{2-h-j+k}}{\alpha^{1+k-h} \beta^{1+k-j}}, \quad (B28)$$

$$\rho_{hjk}^J(s) = \frac{(-1)^k 4^{-k-2}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(4-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{2-h-j+k} (2m^2(\alpha + \beta) + \alpha\beta s(h+j-k-5))}{\alpha^{1+k-h} \beta^{1+k-j}}, \quad (B29)$$

where $h, j, k > 0$, $h + j - k \leq 2$.

$$\rho_{hjk}^K(s) = \frac{(-1)^k 2^{-2k-3}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(3-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{1-h-j+k} (2m^2(\alpha + \beta) + \alpha\beta s(h + j - k - 4))}{\alpha^{1+k-h} \beta^{k-j}}, \quad (B30)$$

where $h, j, k > 0$, and $h + j - k \leq 1$.

$$\rho_{hjk}^L(s) = \frac{(-1)^k 4^{-k-1}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(3-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{-h-j+k}}{\alpha^{k-h} \beta^{k-j}} [6(m^2(\alpha + \beta) - \alpha\beta s)^2 - \alpha\beta s(6(m^2(\alpha + \beta) - \alpha\beta s)(2 + k - h - j) - \alpha\beta s(2 + k - h - j)(1 + k - h - j))], \quad (B31)$$

where, $h, j, k > 0$, and $h + j - k \leq 0$.

$$\rho_{144}^L(s) = \frac{3}{512\pi^2 \Gamma(4)^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^3 (m^2(\alpha + \beta) - 2\alpha\beta s)}{\alpha^3} - \frac{m^4}{\pi^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \frac{\alpha s^2 (m^2 - (1 - \alpha)\alpha s)^3}{36864(m^2 - \alpha s)^6}, \quad (B32)$$

$$\rho_{hjk}^N(s) = \frac{(-1)^{k+1} 4^{-k-2}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(4-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{-h-j+k+1}}{\alpha^{k+1-h} \beta^{k+1-j}} \left(2(6\alpha - 1)(\alpha\beta s - m^2(\alpha + \beta))^2 - \alpha\beta s(3 - h - j + k)(\alpha\beta s(2\alpha(h + j - k - 8) + 1) + (12\alpha - 1)m^2(\alpha + \beta)) \right), \quad (B33)$$

where $h, j, k > 0$, $h + j - k \leq 1$.

$$\rho_{hjk}^O(s) = \frac{(-1)^{k+1} 2^{-2k-3}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(4-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{-h-j+k}}{\alpha^{k-h} \beta^{k-j+1}} [6(8\alpha - 3)(m^2(\alpha + \beta) - \alpha\beta s)^3 - 18(4\alpha - 1)\alpha\beta s(3 - h - j + k)(m^2(\alpha + \beta) - \alpha\beta s)^2 + 3(8\alpha - 1)\alpha^2\beta^2 s^2(2 - h - j + k)(3 - h - j + k)(m^2(\alpha + \beta) - \alpha\beta s) - 2\alpha^4 s^3(1 - h - j + k)(2 - h - j + k)(1 - h - j + k)], \quad (B34)$$

where $h, j, k > 0$, $h + j - k \leq 1$.

$$\rho_{144}^O(s) = -\frac{m^4}{36864\pi^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \frac{\alpha^3 s^3 (m^2 - (1 - \alpha)\alpha s)^3}{(m^2 - \alpha s)^6} - \frac{1}{2048\pi^2 \Gamma(4)^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^3 ((8\alpha - 3)m^4(\alpha + \beta)^2 - 4\alpha(10\alpha - 3)\beta m^2 s(\alpha + \beta) + 10\alpha^2(4\alpha - 1)\beta^2 s^2)}{\alpha^3 \beta}, \quad (B35)$$

$$\rho_{144}^O(s) = \frac{1}{36864\pi^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \frac{\alpha^3 s^3 (m^2 - (1 - \alpha)\alpha s)^3}{m^2 (m^2 - \alpha s)^3} - \frac{1}{2048\pi^2 \Gamma(4)^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^3 ((8\alpha - 3)m^4(\alpha + \beta)^2 - 4\alpha(10\alpha - 3)\beta m^2 s(\alpha + \beta) + 10\alpha^2(4\alpha - 1)\beta^2 s^2)}{\beta^4}. \quad (B36)$$

$$\rho_{hjk}^H(s) = \frac{(-1)^{k+1} 2^{-2k-3}}{\pi^2 \Gamma(h) \Gamma(j) \Gamma(k) \Gamma(3-h-j+k)} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \int_{\beta_{min}}^{\beta_{max}} d\beta \frac{(\alpha + \beta - 1)^{k-1} (m^2(\alpha + \beta) - \alpha\beta s)^{-h-j+k-1}}{\alpha^{k-h} \beta^{k-j} (1 - \alpha)^2} \left(12(1 - \alpha)^3(8\alpha - 1)(m^2(\alpha + \beta) - \alpha\beta s)^3 - 3(48\alpha^3 - 100\alpha^2 + 56\alpha - 3)\alpha\beta s(-h - j + k + 2)(\alpha\beta s - m^2(\alpha + \beta))^2 - 2(\alpha - 1)^2(24\alpha - 1)\alpha^2\beta^2 s^2(h + j - k - 2)(h + j - k - 1)(\alpha\beta s - m^2(\alpha + \beta)) + 4(\alpha - 1)^2\alpha^4\beta^3 s^3(h + j - k - 2)(h + j - k - 1)(h + j - k) \right), \quad (B37)$$

where $h, j, k > 0$, $h + j - k \leq -1$.

$$\rho_{110}^I(s) = -\frac{1}{16\pi^2} \sqrt{1 - \frac{4m^2}{s}}, \quad (\text{B38})$$

$$\rho_{120}^I(s) = \frac{1}{16\pi^2} \frac{1}{\sqrt{s(s-4m^2)}}, \quad (\text{B39})$$

$$\rho_{110}^Q(s) = -\frac{1}{16\pi^2} \left(\frac{s}{2} - m^2 \right) \sqrt{1 - \frac{4m^2}{s}}, \quad (\text{B40})$$

$$\rho_{120}^Q(s) = -\frac{1}{16\pi^2} \left(\frac{s}{2} - m^2 \right)^2 \sqrt{1 - \frac{4m^2}{s}}, \quad (\text{B41})$$

$$\rho_{110}^K(s) = -\frac{1}{8\pi^2} \left(\frac{1}{4(1-s/4m^2)} + \sqrt{1 - \frac{4m^2}{s}} \right), \quad (\text{B42})$$

$$\rho_{110}^N(s) = -\frac{2}{32\pi^2} \left((3s-2m^2) \sqrt{1 - \frac{4m^2}{s}} - \frac{4m^4}{\sqrt{s(s-4m^2)}} \right), \quad (\text{B43})$$

$$\rho_{110}^P(s) = \frac{-83m^6 - 559m^4s + 326m^2s^2 - 53s^3}{120\pi^2 s \sqrt{s(s-4m^2)}}, \quad (\text{B44})$$

$$\Pi^I(M_B^2) = \frac{1}{4\pi^2} \int_0^1 dx \frac{m^2}{x^2 M_B^2} \exp \left[-\frac{m^2}{x(1-x)M_B^2} \right], \quad (\text{B45})$$

$$\Pi^{II}(M_B^2) = -\frac{1}{8\pi^2} \int_0^1 dx \frac{m^6}{x(1-x)M_B^2} \exp \left[-\frac{m^2}{x(1-x)M_B^2} \right]. \quad (\text{B46})$$

The integration limits appears in the above expression are:

$$\alpha_{max} = \frac{1 + \sqrt{1 - 4m^2/s}}{2} \quad \alpha_{min} = \frac{1 - \sqrt{1 - 4m^2/s}}{2} \quad (\text{B47})$$

$$\beta_{max} = 1 - \alpha \quad \beta_{min} = \frac{\alpha m^2}{\alpha s - m^2}. \quad (\text{B48})$$